Exploiting gauge and constraint freedom in hyperbolic formulations of Einstein's equations

Olivier Sarbach* and Manuel Tiglio†
Department of Physics and Astronomy, Louisiana State University,
202 Nicholson Hall, Baton Rouge, LA 70803-4001

We present new many-parameter families of strongly and symmetric hyperbolic formulations of Einstein's equations that include quite general algebraic and live gauge conditions for the lapse.

The first system that we present has 30 variables and incorporates an algebraic relationship between the lapse and the determinant of the three metric that generalizes the densitized lapse prescription. The second system has 34 variables and uses a family of live gauges that generalizes the Bona-Masso slicing conditions.

These systems have free parameters even after imposing hyperbolicity and are expected to be useful in 3D numerical evolutions. We discuss under what conditions there are no superluminal characteristic speeds.

I. INTRODUCTION

The use of hyperbolic formulations of Einstein's equations in numerical simulations has several advantages (see [1] for reviews). In particular this includes the ability of giving boundary conditions that are consistent with both the evolution equations and the constraints [2]. At the discrete level well posedness supplemented by consistency —in order to make sure one is solving Einstein's equations— implies that one can find discretizations that are numerically stable in the sense of the Lax theorem, a fact that is otherwise not necessarily true [3].

Well posedness guarantees that there is a bound in the growth of the solutions that is independent of the initial data, but it does allow this bound to grow with time, and even fast. If this happens, numerical solutions can grow fast as well and, since Einstein's equations are non-linear, this can make the code crash in a finite time. This is what usually happens in 3D black hole simulations. This growth might appear due to several factors, including fast growing gauge or constraint violating modes.

Initially, hyperbolic formulations of Einstein's equations with associated well posed initial-value problems relied on the use of harmonic coordinates [4], that is, $\nabla^\alpha\nabla_\alpha x^\mu = 0$. Although these coordinates have been used recently (see [5] and references therein, and also [6]), they might be too restrictive for numerical applications, since one might want to use another gauge which is better suited to the dynamics of a given numerical evolution. Initial efforts towards relaxing the harmonic condition introduced hyperbolic formulations that used time harmonic coordinates, i.e. $\nabla^\alpha\nabla_\alpha t = 0$ (see [1, 7] and references therein). This condition can be written as

$$\partial_t N - \beta^i\partial_i N = -N^2 K \ ,$$

where K is the trace of the extrinsic curvature, β^i the shift vector, and N the lapse. A generalization of the time harmonic slicing, of the form

$$\partial_t N - \beta^i\partial_i N = -N^2 f(N) K \tag{1}$$

with $f > 0$ was introduced by Bona and Masso (BM) [8] [20], being able to incorporate this condition in a strongly hyperbolic (SH) evolution system. Here by SH we mean that the principal part has a complete set of eigenvectors with real eigenvalues. If the principal part can be diagonalized with a transformation that is uniformly bounded and smooth in all of its arguments, then the initial value problem can be shown to be well posed (this will be discussed later). The BM formulation represents the first effort in moving away from the time harmonic slice while achieving strong hyperbolicity, incorporating other gauge conditions often used in numerical relativity. For example, (1) includes the "$1 + \log$" condition ($f = a/N$, with a some constant). In fact, given a hyperbolic formulation of Einstein's equations that introduces the lapse as a dynamical variable through condition (1), one can add to the right hand side (RHS) of this equation any function S of spacetime,

$$\partial_t N - \beta^i\partial_i N = -N^2 f(N) K + S(x^\mu) \ , \tag{2}$$

*Electronic address: sarbach@phys.lsu.edu
†Electronic address: tiglio@lsu.edu

without changing the level of hyperbolicity of the whole system. From this observation it is clear that any spacetime with given lapse and shift satisfies equation (2), provided S is chosen appropriately (namely, $S = \partial_t N - \beta^i \partial_i N + N^2 f(N) K$).

A closely related slicing condition, which has also been used in several hyperbolic formulations (see [1, 7, 10] and references therein), is obtained by densitizing the lapse. That is, one writes

$$N = e^Q g^\sigma \ , \tag{3}$$

where $Q = Q(x^\mu)$ is an arbitrary but a priori specified function of spacetime, g is the determinant of the three-metric and σ any constant (strong hyperbolicity implies $\sigma > 0$). As before, given any spacetime with lapse and shift there is always a Q that satisfies eq.(3). Restricting to the case of zero shift and taking a time derivative of both sides of Eq.(3) one gets

$$\partial_t N = -2\sigma N^2 K + N \partial_t Q \ ,$$

which shows that in this case condition (3) is time harmonic if $\sigma = 1/2$ and $\partial_t Q = 0$.

Most hyperbolic formulations use either a densitized lapse or some of the BM conditions. The Kidder-Scheel–Teukolsky (KST) formulation [10], for example, uses densitized lapse. This formulation is a many-parameter generalization of previous systems [11] that is SH or symmetric hyperbolic if these free parameters satisfy certain inequalities. KST have shown that one can make use of the freedom in these parameters to extend the lifetime of 3D numerical evolutions of black holes.

As already discussed, one can always evolve a known spacetime in a given slicing by using the appropriate densitization Q. Sometimes one can even use the same function Q to evolve a high, non-linear distortion of the same spacetime [12]. But it is not clear what densitization to choose if one aims to evolve a spacetime that describes, say, the collision of two black holes. This is a rather general problem of gauge prescription. For example, even though there are some features of the BM slicings that are already understood (like its singularity avoiding properties), it is still far from clear what the "most appropriate choice" for the function $f(N)$ is (see section V). However, it seems one could benefit from more general slicing conditions in the same way the free parameters of the KST system turn out to be useful even though one does not completely understand why.

The aim of this paper is to present families of SH and symmetric formulations of Einstein's equations that combine both the freedom of the KST system with a family of lapse choices that includes those described above.

The first family that we present is a straightforward generalization of the KST one that has, instead of a densitized lapse, an algebraic relation of the form

$$N = N(g, x^\mu) \ . \tag{4}$$

That is, the lapse still depends on the spacetime coordinates and the determinant of the three-metric in an a priori prescribed way, but now the power law dependency on g is relaxed. The motivation for introducing this generalization is that it is closely related to the BM conditions, in the same way densitized lapse is related to a time harmonic slicing. More specifically, condition (4) is equivalent to a particular case of condition (5) presented below provided the function F in Eq.(5) satisfies $\partial_K F > 0$ or, equivalently, $\partial_g N > 0$. As we will see later, these two conditions are necessary for strong hyperbolicity. Therefore, in those cases there will be a correspondence between the live gauges and their algebraic counterpart (though the resulting evolution equations will not necessarily be equivalent off the constraint surface). This family has, as the KST one, 30 variables and is presented in Section II. There we also perform a characteristic analysis in order to give sufficient conditions for strong hyperbolicity to hold and in order to analyze under what conditions there are no superluminal speeds.

In section III we present a many-parameter family of SH formulations that includes the following live gauge conditions

$$\partial_t N - \beta^i \partial_i N = -N F(N, K, x^\mu) \ , \tag{5}$$

where F is any function of its arguments that satisfies $\partial_K F > 0$. These are generalizations of the BM ones that relax the linear dependency on K. Since now there are evolution equations for the lapse and its spatial derivatives, our system has 34 variables. This system differs from the BM formulation in various aspects. For example, it uses different variables (in fact, the BM formulation introduces three extra variables in addition to the ones we consider here).

In section IV we will explicitly show that some of the formulations presented in this paper are symmetric hyperbolic; well posedness of the initial value problem follows immediately for these cases. These symmetric hyperbolic systems are obtained by restricting some of the free parameters of our formulations, but not the gauge choice. That is, we are able to get symmetric hyperbolic systems with lapses given by Eqs. (4) or (5).

II. STRONGLY HYPERBOLIC FORMULATIONS WITH AN ALGEBRAIC GAUGE

The system presented in this section consists of the 30 variables $\{g_{ij}, K_{ij}, d_{kij}\}$, where g_{ij} is the three-metric, K_{ij} the extrinsic curvature, and where the extra variables d_{kij} are equal to the first order spatial derivatives $\partial_k g_{ij}$ of the three-metric provided the constraints are satisfied.

The evolution equations in vacuum are obtained by adding constraints to the RHS of the evolution equations obtained from setting to zero the four-dimensional Ricci tensor. Following the notation of KST (except that here we define $\partial_0 \equiv (\partial_t - \pounds_\beta)/N$),

$$\begin{aligned}
\partial_0 g_{ij} &= -2K_{ij} \,, &(6)\\
\partial_0 K_{ij} &= R_{ij} - \frac{1}{N}\nabla_i\nabla_j N - 2K_{ia}K^a_{\ j} + KK_{ij} + \gamma\, g_{ij}C + \zeta\, g^{ab}C_{a(ij)b} \,, &(7)\\
\partial_0 d_{kij} &= -2\partial_k K_{ij} - 2\frac{\partial_k N}{N}K_{ij} + \eta\, g_{k(i}C_{j)} + \chi\, g_{ij}C_k \,, &(8)
\end{aligned}$$

where $\{\gamma, \zeta, \eta, \chi\}$ are free parameters, $C = (R - K_{ab}K^{ab} + K^2)/2$ is the Hamiltonian constraint, $C_i = \nabla^a K_{ai} - \nabla_i K$ the momentum one, and $C_{kij} = d_{kij} - \partial_k g_{ij}$, $C_{lkij} = \partial_{[l}d_{k]ij}$ are constraints that arise due to the introduction of the extra variables. The Ricci tensor R_{ij} belonging to the three-metric is written as

$$R_{ij} = \frac{1}{2}g^{ab}\left(-\partial_a d_{bij} + \partial_a d_{(ij)b} + \partial_{(i}d_{|ab|j)} - \partial_{(i}d_{j)ab}\right) + \frac{1}{2}d_i^{\ ab}d_{jab} + \frac{1}{2}(d_k - 2b_k)\Gamma^k_{\ ij} - \Gamma^k_{\ lj}\Gamma^l_{\ ik} \,,$$

where $b_j \equiv d_{kij}g^{ki}$, $d_k \equiv d_{kij}g^{ij}$ and

$$\Gamma^k_{\ ij} = \frac{1}{2}g^{kl}\left(2d_{(ij)l} - d_{lij}\right) .$$

Similarly, the momentum constraint gives

$$C_i = g^{ab}\left(\partial_a K_{bi} - \partial_i K_{ab}\right) + \frac{1}{2}(d^k - 2b^k)K_{ki} + \frac{1}{2}d_i^{\ ab}K_{ab} \,.$$

The shift is assumed to be a prescribed function of spacetime. In contrast to KST, where the lapse is defined by Eq. (3), here we will consider the lapse to be an arbitrary function of the coordinates and the determinant of the three metric, $N = N(g, x^\mu)$ (as described below, strong hyperbolicity requires $\partial_g N > 0$). In this case, we have

$$\begin{aligned}
\frac{\partial_k N}{N} &= \sigma_{eff}d_k + \frac{\partial N}{N\partial x^k} \,,\\
\frac{1}{N}\nabla_i\nabla_j N &= \sigma_{eff}\left(g^{ab}\partial_{(i}d_{j)ab} - d_i^{\ ab}d_{jab} + d_i d_j\right) + \frac{1}{N}\left(\frac{\partial^2 N}{\partial g^2}g^2 d_i d_j + g\frac{\partial^2 N}{\partial g\partial x^i}d_j + g\frac{\partial^2 N}{\partial g \partial x^j}d_i + \frac{\partial^2 N}{\partial x^i\partial x^j}\right)\\
&\quad - \Gamma^k_{\ ij}\left(\sigma_{eff}d_k + \frac{\partial N}{N\partial x^k}\right),
\end{aligned}$$

where the "effective" σ is defined by

$$\sigma_{eff} := gN^{-1}\frac{\partial N}{\partial g}$$

(σ_{eff} coincides with σ in Eq.(3) if the lapse is densitized). In order to analyze hyperbolicity one has to look at the principal part of the system, that is, the terms that have spatial derivatives of the main variables. In this case the principal part is

$$\begin{aligned}
\partial_0 g_{ij} &= l.o. \,, &(9)\\
\partial_0 K_{ij} &= \frac{1}{2}g^{ab}\left(-\partial_a d_{bij} + (1+\zeta)\partial_a d_{(ij)b} + (1-\zeta)\partial_{(i}d_{|ab|j)} - (1+2\sigma_{eff})\,\partial_{(i}d_{j)ab} + \gamma\, g_{ij}g^{kl}\partial_a(d_{klb} - d_{bkl})\right) + l.o. \,, &(10)\\
\partial_0 d_{kij} &= -2\partial_k K_{ij} + \eta\, g_{k(i}g^{ab}(\partial_{|a|}K_{j)b} - \partial_{j)}K_{ab}) + \chi\, g_{ij}g^{ab}(\partial_a K_{kb} - \partial_k K_{ab}) + l.o. \,, &(11)
\end{aligned}$$

where $l.o.$ stands for "lower order terms". The characteristic speeds in the direction n^i are given by $\beta^i n_i$, $\pm N + \beta^i n_i$, $\pm N\sqrt{\lambda_i} + \beta^i n_i$ (see next subsection for a derivation and details), where

$$\begin{aligned}
\lambda_1 &= 2\sigma_{eff} \,,\\
\lambda_2 &= 1 + \chi - \frac{1}{2}(1+\zeta)\eta + \gamma(2 - \eta + 2\chi) \,,\\
\lambda_3 &= \frac{1}{2}\chi + \frac{3}{8}(1-\zeta)\eta - \frac{1}{4}(1+2\sigma_{eff})(\eta + 3\chi) \,.
\end{aligned}$$

The system is SH if

$$\lambda_j > 0, \quad \text{for } j = 1, 2, 3,$$
$$\lambda_3 = \frac{1}{4}(3\lambda_1 + 1) \quad \text{if } \lambda_1 = \lambda_2.$$

The system has no superluminal speeds provided that $0 \le \lambda_i \le 1$. One way of achieving this is by asking all the λ_i's to be 0 or 1. Since $\lambda_i > 0$ for strong hyperbolicity, one needs $\lambda_i = 1$. In particular, the condition $\lambda_1 = 1$ is possible only if the lapse is densitized with a constant σ_{eff} equal to 1/2. In that case one has two families of formulations with speeds along the light cone or normal to the hypersurfaces, one of them bi-parametric and the other one mono-parametric, see [10]. If σ_{eff} is not constant, one can ask all the speeds but λ_1 to be one. If one requires the parameters $\zeta, \gamma, \eta, \chi$ to be constant this leads to two mono-parametric families of SH systems:

$$\sigma_{eff} > 0\,, \quad \zeta = -\frac{5}{3}, \gamma \text{ arbitrary }, \eta = \frac{6}{5}, \chi = -\frac{2}{5}$$

or

$$\sigma_{eff} > 0\,, \quad \zeta = -\frac{5\chi + 8}{9\chi}, \gamma = -\frac{2}{3}, \eta = -3\chi, \chi \neq 0 \text{ arbitrary}$$

As shown below, the characteristic modes (and whether they are superluminal or not) associated with λ_1 are independent of the formulation, they depend only on the choice of the slicing condition.

A. Characteristic analysis

Here, we discuss under which conditions the system (9-11) is SH, and what the characteristic speeds are. In order to do so, we choose a fixed direction n^i with $g_{ij}n^in^j = 1$ and study the eigenvalues of the principal part of (9-11) in the direction of n^k:

$$\mu \begin{pmatrix} g_{ij} \\ K_{ij} \\ d_{kij} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{A} \\ 0 & \boldsymbol{B} & 0 \end{pmatrix} \begin{pmatrix} g_{ij} \\ K_{ij} \\ d_{kij} \end{pmatrix}, \tag{12}$$

where the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are obtained from the principal parts of (10) and (11), respectively, by replacing ∂_k by n_k. The characteristic speeds of the system are given by

$$N\mu + \beta^i n_i\,,$$

where μ are the eigenvalues of problem (12). The system is SH if all the eigenvalues are real and the corresponding eigenvectors are complete.

These eigenvalues are either zero or can be obtained by considering the eigenvalue problem

$$\mu^2 K_{ij} = \boldsymbol{AB} K_{ij}\,. \tag{13}$$

Explicitly, we have

$$\mu^2 K_{ij} = K_{ij} + A n_{(i} n^s K_{j)s} + B n_i n_j K + C g_{ij}\,(n^r n^s K_{rs} - K)\,,$$

where the coefficients A, B and C are

$$\begin{aligned} A &= -2 + \chi - \frac{3}{4}(\zeta - 1)\eta - \left(\sigma_{eff} + \frac{1}{2}\right)(\eta + 3\chi)\,, \\ B &= -\chi + \frac{3}{4}(\zeta - 1)\eta + \left(\sigma_{eff} + \frac{1}{2}\right)(2 + \eta + 3\chi)\,, \\ -2C &= \chi - \frac{1}{2}(\zeta + 1)\eta + \gamma(2 - \eta + 2\chi). \end{aligned}$$

Next, we complete n^i to a complex orthonormal basis n^i, m^i, $\bar{m}^i$ such that $n^in^jg_{ij} = 1$, $m^i\bar{m}^jg_{ij} = 1$, and all other scalar products are zero. We then decompose K_{ij} according to

$$K_{ij} = a\, n_i n_j + b\, m_{(i}\bar{m}_{j)} + \left[c\, n_{(i} m_{j)} + d\, m_{(i} m_{j)} + c.c.\right],$$

where a, b are real and c, d are complex. In this basis the linear operator $\boldsymbol{AB}$ takes the simple form

$$\boldsymbol{AB} = \begin{pmatrix} 1+A+B & B-C & 0 & 0 \\ 0 & 1-2C & 0 & 0 \\ 0 & 0 & 1+\frac{A}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

The eigenvalues of this matrix are

$$\begin{aligned} \lambda_1 &= 1+A+B = 2\sigma_{eff}, \\ \lambda_2 &= 1-2C = 1+\chi-\frac{1}{2}(1+\zeta)\eta+\gamma(2-\eta+2\chi), \\ \lambda_3 &= 1+\frac{A}{2} = \frac{1}{2}\chi+\frac{3}{8}(1-\zeta)\eta-\frac{1}{4}(1+2\sigma)(\eta+3\chi), \\ \lambda_4 &= 1, \end{aligned}$$

and we have

$$B-C = \lambda_1+\frac{1}{2}(\lambda_2+1)-2\lambda_3\,.$$

We demand that the matrix $\boldsymbol{AB}$ be diagonalizable and have only positive real eigenvalues. As we show now, this is a sufficient condition for the original system (9-11) to be SH.

First note that since $\boldsymbol{AB}$ is positive definite, $\boldsymbol{B}$ is injective and $\boldsymbol{A}$ is surjective. Now let $K_{ij}^{(1)}, ..., K_{ij}^{(6)}$ denote the six real eigenvectors of $\boldsymbol{AB}$ which have the eigenvalues $\omega^{(1)}, ..., \omega^{(6)}$. Then, the 12 vectors

$$v_{\pm s} = \begin{pmatrix} 0 \\ \pm\sqrt{\omega^{(s)}}K_{ij}^{(s)} \\ (\boldsymbol{B}K^{(s)})_{kij} \end{pmatrix}, \tag{15}$$

are eigenvectors of (12) with eigenvalues $\mu_{\pm s} = \pm\sqrt{\omega^{(s)}}$, $s = 1, 2, ..., 6$. These vectors are linearly independent since $\omega^{(s)} \neq 0$ and since $\boldsymbol{B}$ is injective. The remaining eigenvectors have $\mu = 0$. Since $\boldsymbol{A}$ is surjective, we have

$$\dim\ker\boldsymbol{A} = 18 - \dim\Im\boldsymbol{A} = 12,$$

thus there are 18 zero eigenvectors, 12 with non-trivial d_{kij}'s which lie in the kernel of $\boldsymbol{A}$, and 6 with non-trivial g_{ij}'s. Therefore, we have a set of 30 independent eigenvectors with real eigenvalues. In order to show that the system (9-11) yields a well posed initial-value problem, one has to look at the matrix $S(n^i)$ whose columns are the 30 eigenvectors of the principal part of the system and show that $S(n^i)$ and its inverse are uniformly bounded and that they depend smoothly on n^i and the metric coefficients g_{ij} and g^{ij}. We do not show this. However, if we linearize the equations around flat spacetime in Cartesian coordinates, the principal part depends only on n^i and the flat metric δ_{ij}. Using an isotropy argument it is not difficult to show that in this case the matrix $S(n^i)$ can be obtained from $S(n_0^i)$ in a fixed direction n_0^i by a rotation which maps n_0^i to n^i. It follows from this that $S(n^i)$ can be chosen such that its norm and the norm of its inverse are uniformly bounded for all n^i with $\delta_{ij}n^in^j = 1$. In this case, this is sufficient to guarantee well posedness [13].

The matrix $\boldsymbol{AB}$ is diagonalizable if and only if $\lambda_1, \lambda_2, \lambda_3 > 0$ and $\lambda_3 = (3\lambda_1+1)/4$ whenever $\lambda_1 = \lambda_2$.

B. Characterization of the eigenmodes

Here we give an interpretation to the eigenmodes of (13) when the SH field equations (6,7,8) are linearized around Minkowski spacetime. One can see that in such case the characteristic modes are solutions to Einstein's equations. According to (14) these modes are given by the following vectors corresponding to the values of (a, b, c, d):

$$\begin{aligned} \lambda = 1: &\quad (0,0,0,1), (0,0,0,i), \\ \lambda = \lambda_3: &\quad (0,0,1,0), (0,0,i,0), \\ \lambda = \lambda_2: &\quad (0,1,0,0) \text{ if } \lambda_2 = \lambda_1, \quad (C-B, 2\sigma_{eff}-1+2C, 0, 0) \text{ otherwise }, \\ \lambda = \lambda_1: &\quad (1,0,0,0). \end{aligned} \tag{16}$$

The first two modes, which propagate along the light cone (ie. which have speeds $\pm N + \beta^i n_i$) are physical modes: By making a Fourier transform of the linearized Hamiltonian and momentum constraints, it is easy to check that the constraints are satisfied by the corresponding 4 eigenmodes (15). Next, we have six constraint violating modes which have characteristic speeds $\pm N\sqrt{\lambda_3} + \beta^i n_i$, $\pm N\sqrt{\lambda_2} + \beta^i n_i$: Indeed, the Fourier transformed linearized momentum constraint yields

$$n^j K_{ij} - n_i K = -b\, n_i + \frac{1}{2}\, c\, m_i + \frac{1}{2}\, \bar{c}\, \bar{m}_i \,,$$

and only modes with $b = c = 0$ satisfy these constraints. The modes with characteristic speeds $\pm N\sqrt{2\sigma_{eff}} + \beta^i n_i$ are gauge modes: With respect to an infinitesimal coordinate transformation of the form $\delta x^\mu \mapsto \delta x^\mu + f \delta_t^\mu$, the extrinsic curvature transforms according to

$$K_{ij} \mapsto K_{ij} + n_i n_j f \,,$$

therefore

$$a \mapsto a + f \,,$$

and the eigenmode (16) with eigenvalue $\lambda_1 = 2\sigma_{eff}$ can be gauged away.

III. STRONGLY HYPERBOLIC FORMULATIONS WITH A LIVE GAUGE

Here we construct SH formulations with live gauges. In addition to the variables of the previous section, we promote the lapse N and three extra quantities A_i which are equal to $(\partial_i N)/N$ if the constraints are satisfied, to independent variables. Our variables are, therefore, $\{g_{ij}, K_{ij}, d_{kij}, N, A_i\}$. As in the previous section, the shift is assumed to be an arbitrary but apriori prescribed function of spacetime.

As evolution equation for the lapse we consider

$$\partial_0 N = -F(N, K, x^\mu),$$

where $F(N, K, x^\mu)$ is an arbitrary function of its arguments. An evolution equation for A_i is obtained from this by taking a spatial derivative:

$$\partial_0 A_i = -\frac{\partial F}{\partial N} A_i - \frac{1}{N}\frac{\partial F}{\partial K}\partial_i K - \frac{1}{N}\frac{\partial F}{\partial x^i} + \xi\, C_i \,,$$

where we have also added the momentum constraint with a free parameter ξ.

The evolution equations for g_{ij}, K_{ij} and d_{kij} are the same as in (6-8) where now

$$\frac{1}{N}\nabla_i\nabla_j N = \partial_{(i}A_{j)} - \Gamma^k{}_{ij} A_k + A_i A_j \,,$$

and where we replace $(\partial k N)/N$ by A_k in the evolution equation for d_{kij}.

The principal part of the system is

$$\begin{aligned}
\partial_0 g_{ij} &= l.o.\,, \\
\partial_0 K_{ij} &= \frac{1}{2} g^{ab}\left(-\partial_a d_{bij} + (1+\zeta)\partial_a d_{(ij)b} + (1-\zeta)\partial_{(i} d_{|ab|j)} - \partial_{(i} d_{j)ab} + \gamma\, g_{ij} g^{kl}\partial_a(d_{klb} - d_{bkl})\right) - \partial_{(i}A_{j)} + l.o.\,, \\
\partial_0 d_{kij} &= -2\partial_k K_{ij} + \eta\, g_{k(i} g^{ab}(\partial_{|a|}K_{j)b} - \partial_{j)}K_{ab}) + \chi\, g_{ij} g^{ab}(\partial_a K_{kb} - \partial_k K_{ab}) + l.o.\,, \\
\partial_0 N &= l.o.\,, \\
\partial_0 A_i &= -\frac{1}{N}\frac{\partial F}{\partial K}\partial_i K + \xi\, g^{ab}(\partial_a K_{bi} - \partial_i K_{ab}) + l.o.\,.
\end{aligned}$$

In order to get the conditions under which the system is SH, and in order to get the characteristic speeds, we can use the techniques of the previous section: The principal part has the form

$$\mu \begin{pmatrix} N \\ g_{ij} \\ K_{ij} \\ u \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{A} \\ 0 & 0 & \boldsymbol{B} & 0 \end{pmatrix} \begin{pmatrix} N \\ g_{ij} \\ K_{ij} \\ u \end{pmatrix},$$

where $u = (d_{kij}, A_i)^T$. Explicitly writing the matrix $\boldsymbol{AB}$, we obtain the same result as in (14) except that now

$$\begin{aligned}
A &= -2 - \frac{1}{2}\chi - \frac{1}{4}(3\zeta - 1)\eta - \xi \\
B &= \frac{1}{2}\chi + \frac{1}{4}(3\zeta - 1)\eta + \xi + 1 + 2\sigma_{eff} \\
-2C &= \chi - \frac{1}{2}(\zeta + 1)\eta + \gamma(2 - \eta + 2\chi)\,,
\end{aligned}$$

where $\sigma_{eff} = (\partial_K F)/(2N)$. Therefore, the characteristic speeds of the system are $\beta^i n_i$, $\pm N + \beta^i n_i$, $\pm N\sqrt{\lambda_i} + \beta^i n_i$, with

$$\begin{aligned}
\lambda_1 &= 2\sigma_{eff}\,, \\
\lambda_2 &= 1 + \chi - \frac{1}{2}(1 + \zeta)\eta + \gamma(2 - \eta + 2\chi), \\
\lambda_3 &= -\frac{1}{4}\chi - \frac{1}{8}(3\zeta - 1)\eta - \frac{1}{2}\xi\,.
\end{aligned}$$

In particular, if one chooses $\xi = \sigma_{eff}(\eta + 3\chi)$, these speeds are exactly those of the system of the previous section and strong hyperbolicity holds under the same conditions. The conditions under which there are no superluminal speeds are also the same, and the mode associated with λ_1 is related to the choice of gauge as well. More generally, one can, as in the previous section, ask all the λ_i's but λ_1 to be 1. This leads to two SH many-parameter families, one with three free parameters (γ, ζ, η):

$$\begin{aligned}
\sigma_{eff} &> 0\,, \\
\gamma &\neq -\frac{1}{2}\,, \\
\chi &= \frac{(1 + \zeta)\eta - 2\gamma(2 - \eta)}{2(1 + 2\gamma)}\,, \\
\xi &= -\frac{1}{2}\chi - \frac{1}{4}(3\zeta - 1)\eta - 2\,,
\end{aligned}$$

and another one with two free parameters (ζ, χ):

$$\sigma_{eff} > 0\,, \quad \gamma = -\frac{1}{2}\,, \quad \zeta\eta = -2\,, \quad \xi = -\frac{1}{2}\chi + \frac{1}{4}\eta - \frac{1}{2}\,.$$

IV. SOME SYMMETRIC HYPERBOLIC SUBFAMILIES

Here we show that some of our formulations are symmetric hyperbolic. We do not intend to give the most general conditions under which this holds but, instead, show that there are at least some subfamilies that are symmetric hyperbolic. We start discussing the family of live gauge conditions and then briefly discuss the algebraic case.

We show that the principal part of the system can be brought into symmetric form by using a transformation which does not depend on n^i. In order to find such a transformation, it is convenient to first transform the variables K_{ij} and d_{kij} into their trace and trace-less parts:

$$\begin{aligned}
K_{ij} &= P_{ij} + \frac{1}{3}\, g_{ij} K, \\
d_{kij} &= 2e_{kij} + \frac{3}{5}\, g_{k(i}\Gamma_{j)} - \frac{1}{5}\, g_{ij}\Gamma_k + \frac{1}{3}\, g_{ij} d_k\,,
\end{aligned}$$

where $\Gamma_k = b_k - d_k/3$ and P_{ij} and e_{kij} are trace-less in all their indices. In terms of the new variables K, A_{ij}, e_{kij}, Γ_k, d_k, the principal part is

$$\begin{aligned}
\mu K &= \left(1 + \frac{3}{2}\gamma\right)\left(\Gamma_n - \frac{2}{3}d_n\right) - A_n\,, \\
\mu P_{ij} &= -e_{nij} + (1 + \zeta)e_{(ij)n} + \left[\frac{1}{20}(5 - 9\zeta)n_{(i}\Gamma_{j)} - \frac{1}{6}\, n_{(i}d_{j)} - n_{(i}A_{j)}\right]^{TF}\,,
\end{aligned}$$

$$\begin{aligned}
\mu\Gamma_k &= \left(\frac{5}{3}\eta - 2\right) P_{kn} - \frac{10}{9}\eta\, n_k K, \\
\mu d_k &= (\eta + 3\chi) P_{kn} - \frac{2}{3}(3 + \eta + 3\chi) n_k K, \\
\mu A_k &= \xi P_{kn} - \left(2\sigma_{eff} + \frac{2}{3}\xi\right) n_k K, \\
\mu e_{kij} &= -n_k P_{ij} + \frac{3}{5}\, g_{k(i} P_{j)n} - \frac{1}{5}\, g_{ij} P_{kn}\,,
\end{aligned}$$

where TF indicates the trace-free part, and where $\Gamma_n = n^k \Gamma_k$, $e_{nij} = n^k e_{kij}$ etc. First, we see that if we set $\zeta = -1$, the term involving e_{kij} in the equation for P_{ij} is the symmetric counterpart of the term involving P_{ij} in the equation for e_{kij}. Now, we can try to find three independent linear combinations of the variables Γ_k, d_k and A_k and rescale K such that the principal part becomes symmetric. For the choice $\zeta = -1$, $\gamma = -2/3$, $\eta = 6m/5$, $\chi = -2m/5$, we find that the transformation

$$K \mapsto c_1 K, \qquad \Gamma_k \mapsto c_2 \left(\Gamma_k - \frac{2m}{3} d_k\right), \qquad d_k \mapsto c_3 \left[\left(\sigma_{eff} + \frac{1}{3}\xi\right) d_k - A_k\right], \qquad A_k \mapsto c_4 A_k\,,$$

yields a symmetric system, provided that

$$m > 1, \qquad \sigma_{eff} > p \equiv \frac{7m}{15} - \frac{1}{6}\,,$$

and if we choose

$$\begin{aligned}
\xi &= \frac{1}{2c_4^2}\left[-(1 + 3c_4^2\sigma_{eff}) + \sqrt{(1 - 3c_4^2\sigma_{eff})^2 + 12c_4^2 p}\right], \\
c_1^2 &= 2c_4^2\left(\sigma_{eff} + \frac{1}{3}\xi\right), \\
c_2^2 &= \frac{7}{20(m-1)}\,, \\
c_3^2 &= -\frac{1 + c_4^2\xi}{\xi}\,.
\end{aligned}$$

Note that the definition of ξ and the requirement $\sigma_{eff} > p$ guarantee that $-1 < c_4^2\xi < 0$ and $3\sigma_{eff} + \xi > 0$. The characteristic speeds of the system are $\beta^i n_i$, $\pm N + \beta^i n_i$, $\pm N\sqrt{\lambda_i} + \beta^i n_i$, with

$$\lambda_1 = 2\sigma_{eff}, \qquad \lambda_2 = 2p, \qquad \lambda_3 = \frac{7m}{10} - \frac{1}{2}\xi.$$

Setting, for example, $m = 15/14$ and $c_4^2 = 2$ yields $\lambda_2 = 2/3$ and $\lambda_3 = 3/4 - \xi/2 < 1$. Therefore, as long as $\sigma_{eff} > 1/3$, the system is symmetric hyperbolic and has no superluminal constraint violating modes.

One can similarly show that there are symmetric hyperbolic subfamilies in the algebraic case. For example, the choice

$$\zeta = -1, \qquad \gamma = -\frac{2}{3} \qquad \eta = \frac{3}{7}(1 + 6\sigma_{eff}), \qquad \chi = -\frac{1}{7}(1 + 6\sigma_{eff}),$$

yields a symmetrizable system provided that $\sigma_{eff} > 3/10$. In this case the λ_i's are $\lambda_1 = \lambda_2 = 2\sigma_{eff}$, $\lambda_3 = (1{+}6\sigma_{eff})/4$.

Finally, as in KST, one could perform a many-parameter change of variables in any of our formulations without affecting the spectrum of the principal part and therefore without changing the level of hyperbolicity.

V. DISCUSSION

The freedom in the KST formulations has proven to be very useful in improving the stability of 3D single black hole numerical evolutions. There are studies underway to understand the reasons behind this, but there is still not a clear picture. If the lifetime of present evolutions using this system carry over to dynamical situations, interesting parts of a binary black hole collision could be described. However, one possible obstacle is the lack of flexibility of the family

of lapses considered in the KST formulation, namely the densitized lapse prescription. On the other hand, the main idea behind the BM formulation is to introduce in SH formulations several of the dynamical slicings that are used in numerical evolutions. The purpose of this paper has been to combine the spirits of the KST and BM formulations, and the result is expected to be useful in 3D evolutions, especially in dynamical ones.

We have considered an algebraic and a live family of slicing conditions that include most of the conditions used in numerical relativity, such as densitized lapse, time harmonic slicings, the "$1 + \log$" case, and the BM conditions. Furthermore, we have shown that one can obtain symmetric hyperbolic systems with these choices of gauges and, therefore, have a well posed initial-value problem. Up to our knowledge, this is the first time this has been achieved. The BM formulation, for example, is shown in [9] to be SH in the sense used in this paper but, as already mentioned, this does not automatically imply well posedness.

Initially the approach to black hole evolutions (see [14] for a review in numerical relativity) was through the use of singularity avoiding (SA) slicings, such as maximal slicing. This gauge condition has been widely used in 1D and 2D, but in 3D several difficulties appear: not only is it computationally expensive to solve elliptic equations at each time step, but also one has to solve this elliptic equation with very good accuracy in order to avoid noise (see, e.g. [15]). This lead to the introduction of live gauges that mimic the maximal condition near the singularity. All of the BM conditions are SA exactly in those cases that lead to SH formulations [8], but there are still differences between different subcases. For example, time harmonic slicings are not as SA [16] as the "$1 + \log$" are, and this is one of the reasons the latter has been used so much. However, if the singularity is not avoided but is excised, SA slicings are in principle no longer needed. On the contrary, these slicings can cause problems since they introduce steep gradients near the horizon. Still, there is evidence that the use of an appropriate shift can help, making the "$1 + \log$" condition useful even in the presence of excision [17].

Having a hyperbolic formulation with no superluminal speeds ($\lambda_i \leq 1$) is an advantage for singularity excision since then one does not need to give boundary conditions at the inner boundary. In the SH families of formulations we have presented in this paper, all the λ_i's but one (λ_1) can apriori be set to 1. As we have shown, the modes associated with λ_1 are gauge modes, and thus, whether these modes are superluminal or not does not depend on the formulation but on the gauge condition considered. In some cases one can decide a priori whether or not $\lambda_1 \leq 1$, but in principle λ_1 might depend on the solution. If that happens, one can ask the code to follow the characteristic speeds and decide during evolution whether boundary conditions at the inner boundary are needed or not (at the outer boundary one has to deal with boundary conditions in any case). If a mode does enter the computational domain and one ignores this fact and continues, for instance, by doing extrapolation, one is implicitly giving boundary conditions that depend on the grid spacing and might not have a consistent limit or be physically correct as resolution is increased. This can indeed happen even with very simple choices of gauge (see, e.g. the appendix of [12]) and the result of such a procedure is uncertain. Even if one is willing to give boundary conditions to modes that enter the domain, the question of what conditions to give still remains. In principle, since physically relevant quantities are gauge-independent, one could give any boundary conditions to these modes. However, what could happen is that a bad choice leads to a gauge the becomes singular after a while. In fact, the same problem arises even in the absence of boundaries when the lapse is not prescribed apriori as a function of spacetime, but, for instance, a live gauge condition is chosen [18].

An issue that we have not addressed in this paper is the introduction of dynamical shifts that may help to follow steep gradients near the horizon or provide some sort of minimal distortion. Current efforts are oriented along this line. Numerical experiments testing the formulations presented in this paper are also underway.

VI. ACKNOWLEDGMENTS

This work was supported in part by NSF grant PHY9800973, the Horace C. Hearne Jr. Institute of Theoretical Physics, the Swiss National Science Foundation, and Fundación Antorchas. We thank Carles Bona, Luis Lehner, Jorge Pullin, and Mark Scheel for useful comments and discussions. During the preparation of this manuscript related work by Alvi has appeared [19].

[1] H. Friedrich and A. Rendall, in *Einstein's Field Equations and their Physical Interpretation*, edited by B. G. Schmidt (Springer, Berlin, 2000), gr-qc/000207; O. Reula "Living Reviews in Relativity", http://www.livingreviews.org.

[2] B. Szilagyi and J. Winicour, gr-qc/0205044; G. Calabrese, L. Lehner, and M. Tiglio, Phys. Rev. D **65**, 104031 (2002); J. Bardeen and L. Buchman, Phys. Rev. D **65**, 064037 (2002); B. Szilagyi, B, Schmidt, and J. Winicour, Phys. Rev. D **65**, 064015 (2002); M. Iriondo and O. Reula, Phys. Rev. D **65**, 044024 (2002); J. M. Stewart, Class. Quantum Grav. **15**, 2865 (1998).

[3] G. Calabrese, J. Pullin, O. Sarbach, and M. Tiglio, *"Convergence and stability in numerical relativity"*, in preparation.

[4] Y. Bruhat, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley, New York, 1962).
[5] D. Garfinkle, Phys. Rev. D **65**, 044029 (2002).
[6] B. Szilagyi and J. Winicour, gr-qc/0205044; B. Szilagyi, B, Schmidt, and J. Winicour, Phys. Rev. D **65**, 064015 (2002).
[7] H. Friedrich, Class. Quant. Grav. **13**, 1451 (1996).
[8] C. Bona *et. al.* Phys. Rev. Lett. **75**, 600 (1995); Phys. Rev. D **56**, 3405 (1997).
[9] C. Bona *et. al.* Phys. Rev. Lett. **68**, 1097 (1992).
[10] L. E. Kidder, M. A. Scheel, and S. A. Teukolsky, Phys. Rev. D **64**, 064017 (2001).
[11] A. Anderson and J. W. York, Jr., Phys. Rev. Lett. **82**, 4384 (1999); S. D. Hern, Ph.D. thesis, University of Cambridge, 1999, gr-qc/0004036; S. Frittelli and O. A. Reula, Phys. Rev. Lett. **76**, 4667 (1996).
[12] G. Calabrese, L. Lehner and M. Tiglio, Phys. Rev. D **65**, 104031 (2002).
[13] H. Kreiss and J. Lorentz, *Initial boundary value problems and the Navier Stokes equation*, Academic Press, San Diego, (1989), Theorem 2.4.1.
[14] L. Lehner, Class. Quant. Grav. **18**, R25 (2001).
[15] P. Anninos *et. al.*, Phys. Rev. D **52**, 2059 (1995).
[16] C. Bona and J. Masso, Phys. Rev. D **38**, 2419 (1988).
[17] M. Alcubierre *et. al.*, Phys. Rev. D **64**, 061501 (2001).
[18] M. Alcubierre, Phys. Rev. D **55**, 5981 (1997); J. Masso and M. Alcubierre, Phys. Rev. D **57**, 4511 (1998).
[19] K. Alvi, gr-qc/0204068.
[20] Actually, there are two BM strongly hyperbolic formulations, the first one uses time harmonic slicing [9], and the second one [8] uses conditions (1). Along this paper, when we refer to the BM formulation, it will be to the second one.